3. REPRESENTATIONS OF FINITE GROUPS

§ 3.1. Representations of Groups

Group theory began with Évariste Galois's study of the solubility of polynomials by radicals. For Galois, a group was a group of permutations on the zeros of a polynomial, but later the subject grew away from polynomials, and groups were considered to be permutation groups on abstract symbols. Finally group theory reached its maturity when it became purely axiomatic and the elements no longer had to be permutations.

The move towards abstraction was certainly a useful one. Yet concrete models can often throw light on abstract structures. A rather trivial, but useful, example is in the area of finite-dimensional vector spaces. A vector was first considered to be an *n*-tuple of scalars such as $(x_1, x_2, \ldots, x_n)$, with addition and scalar multiplication being carried out component-wise. But then vector spaces became fully axiomatised and the *n*-tuples became just one example among many. However, one of the early theorems in linear algebra is that every finite-dimensional vector space is isomorphic to the space of *n*-tuples, for some *n*. So vector spaces can be represented by *n*-tuples and some theorems about abstract vector spaces can be proved by just considering this concrete model.

In group theory we have the theorem that every finite group is isomorphic to a group of permutations, and so we can prove theorems about an abstract group G by considering it as a group of permutations on G. Mostly the extra structure just gets in the way, but the following little theorem shows that the idea is not entirely without merit.

Theorem 1: If $|G| = 2N$, where N is odd, then G has a normal subgroup of order N.

Proof: By Cauchy's Theorem (or even more simply by observing that elements whose order is bigger than 2 come in pairs $\{x, x^{-1}\}$) G has an element of order 2. Regarding G as a group of permutations on G by right multiplication, an element of order 2 would correspond to a permutation of the form (××)(××) … (××), a product of N transpositions. Such a permutation is odd so the map $\theta: G \rightarrow \mathbb{Z}_2$ defined by $g\theta = \begin{cases} 0 \text{ if } g \text{ corresponds to an even permutation} \\ 1 \text{ if } g \text{ corresponds to an odd permutation} \end{cases}$ is a homomorphism whose image is $\mathbb{Z}_2$. Hence $G/\ker\theta \cong \mathbb{Z}_2$ and so $|\ker\theta| = N$. ✋☺

However, representing groups as groups of permutations doesn't appear to be particularly fruitful. By contrast representing groups as groups of matrices is a quite different story. Every permutation on *n* symbols can be represented by an $n \times n$ permutation matrix and so every group of order *n* can be represented by a group of *n*

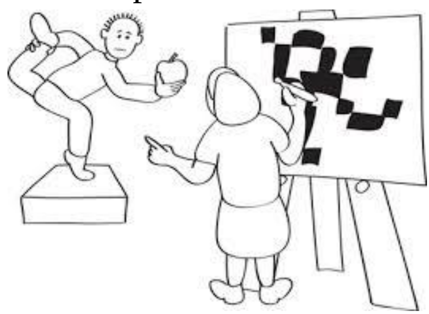
$\times n$ matrices. The advantage of moving across to matrices is that we have the rich theory of linear algebra to make use of. This time we can use the concrete model to prove quite deep theorems in group theory, some of which were, for many years, only proved in this way.

The matrices can be considered to be over any field. Economy might suggest using a small finite field, such as $\mathbb{Z}_2$ (after all the permutation matrices only need 0's and 1's) but the characteristic of the field often gets in the way. The nicest theory uses the field of complex numbers. This is so-called 'classical' representation theory. The field $\mathbb{C}$ has three important properties:

(1) it has characteristic zero – so there's no prime associated with the field;
(2) it's algebraically closed – so eigenvalues exist in the field;
(3) it has the useful structure of the complex conjugate.

Sometimes it's useful to work, not with matrices, but with linear transformations of a vector space, which is essentially the same thing but without having to bother with choosing a specific basis. We start by considering representations of groups by groups of linear transformations over a finite-dimensional vector space over an arbitrary field, but after a while we focus on groups of matrices over the field of complex numbers.

If V is a finite-dimensional vector space over the field F we denote the set of invertible linear transformations from V to V by **Aut(V)**. This is a group under the operation of the product of linear transformations. What is essentially the same thing, is **GL(*n*, F)**, the **general linear group** of degree *n* over the field F, which is the group of invertible $n \times n$ matrices over F.



It's assumed that the reader understands the rudiments of ring theory. We make very little use of that theory except at one point where we assume the Wedderburn Structure Theorem. This is quite a deep theorem that provides a foundation for representation theory. Either you will have spent many weeks elsewhere in proving this theorem or you'll simply have to accept the result without proof.

We've implied that Representation Theory is concerned with matrix groups that are isomorphic to a given one. Actually that's not quite true. We're interested in matrix groups that are *homomorphic images* of a given group, not just isomorphic ones.

A **representation** of **degree** *n* of a group G over the field F is defined to be a group homomorphism ρ: G→ GL(*n*, F). By the first isomorphism theorem the image of a representation ρ is a group of $n \times n$ matrices that's isomorphic to the quotient group G/ker(ρ).

A **linear representation** is a representation of degree 1. This is an important special case. Of course a 1 × 1 matrix behaves like its one and only component so a linear representation is essentially a homomorphism to $F^{\#}$, the group, under multiplication, of the non-zero elements of the field F.

Among the linear representations is the so-called trivial representation. The **trivial representation** is $\tau(g) = 1$ for all $g \in G$. Not very exciting perhaps, but the trivial representation is as important to representation theory as the number 0 is to arithmetic or the empty set to set theory.

Example 1: If $\omega = e^{2\pi i/3}$ the following is a matrix representation of degree 2 for S_3:

I	**(123)**	**(132)**	**(12)**	**(13)**	**(23)**
$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}\omega & 0\\ 0 & \omega^2\end{pmatrix}$	$\begin{pmatrix}\omega^2 & 0\\ 0 & \omega\end{pmatrix}$	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & \omega^2\\ \omega & 0\end{pmatrix}$	$\begin{pmatrix}0 & \omega\\ \omega^2 & 0\end{pmatrix}$

The trivial representation squeezes the group entirely into one element so that no information about the group remains. The kernel of the trivial representation is the whole group. At the other end of the spectrum are the faithful representations. A representation is **faithful** if its kernel is trivial. The image under a faithful representation is isomorphic to the group itself. It might seem that these are the best representations because they don't lose any information. But a suitable collection of 'unfaithful' representations is usually more useful.

Example 2:

The following are some of the representations of the Klein Group V_4 with presentation $\langle A, B \mid A^2, B^2, AB = BA\rangle$.

To begin with there's the trivial representation:
$\tau(1) = 1$, $\tau(A) = 1$, $\tau(B) = 1$, $\tau(AB) = 1$.
Then there are three other linear representations. Since every element of the group satisfies
$g^2 = 1$, a linear representation must map each element to a complex number satisfying $x^2 = 1$. So the linear representations of G are:

	1	A	B	AB
τ	1	1	1	1
α	1	1	−1	−1
β	1	−1	1	−1
γ	1	−1	−1	1

Then there's a faithful representation that maps A to $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ B to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and AB to $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

If G is a group of permutations we can represent each element g by the corresponding permutation matrix, (a_{ij}) where:

$$a_{ij} = \begin{cases} 1 \text{ if } g(i) = j \\ 0 \text{ if } g(i) \neq j \end{cases}$$

Such a representation is called a **permutation representation.** It will always be faithful.

Cayley's theorem shows that every finite group can be considered as a group of permutations on itself since, for $g \in G$, the map $x \to xg$ is a permutation $\pi(g)$ of G and π is a homomorphism. If G has order n we can represent $\pi(g)$ by an $n \times n$ permutation matrix. This permutation representation is called the **regular representation**.

Example 3: The regular representation for V_4 above is:

$$I \to \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, A \to \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&1 \\ 0&0&1&0 \end{pmatrix}, B \to \begin{pmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 1&0&0&0 \\ 0&1&0&0 \end{pmatrix},$$

$$AB \to \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix}.$$

Two representations ρ, σ are **equivalent** if there's an invertible S such that:
$\rho(g) = S^{-1}\sigma(g)\,S$ for all $g \in G$.

Example 4:

The representations of S_3 include:

	I	**(123)**	**(132)**	**(12)**	**(13)**	**(23)**
ρ_1	1	1	1	1	1	1
ρ_2	1	1	1	–1	–1	–1
ρ_3	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & -1\\1 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & 1\\-1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & -1\\0 & -1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\-1 & 1\end{pmatrix}$
ρ_4	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$	$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$
ρ_5	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}\omega & 0\\0 & \omega^2\end{pmatrix}$	$\begin{pmatrix}\omega^2 & 0\\0 & \omega\end{pmatrix}$	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & \omega^2\\\omega & 0\end{pmatrix}$	$\begin{pmatrix}0 & \omega\\\omega^2 & 0\end{pmatrix}$
ρ_6	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$

	I	**(123)**	**(132)**
ρ_7	$\begin{pmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\1&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&1&0&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&1\\0&0&0&1&0&0\\0&0&0&0&1&0\end{pmatrix}$

	(12)	(13)	(23)
ρ_7	$\begin{pmatrix} 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \\ 1&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&1&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&0&1 \\ 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \\ 0&0&1&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&1 \\ 0&0&0&0&1&0 \\ 0&0&0&1&0&0 \\ 0&0&1&0&0&0 \\ 0&1&0&0&0&0 \\ 1&0&0&0&0&0 \end{pmatrix}$

where $\omega = e^{2\pi i/3}$.

- ρ_1, ρ_2 are linear representations; ρ_3, ρ_4 and ρ_5 have degree 2, ρ_6 has degree 3 and ρ_7 has degree 6.
- ρ_1 is the trivial representation.
- ρ_6 and ρ_7 are permutation representations.
- ρ_7 is the regular representation.
- ρ_3, ρ_5, ρ_6 and ρ_7 are faithful.
- ρ_3 is equivalent to ρ_5 since $S^{-1}\rho_3(g)S = \rho_5(g)$ where $S = \begin{pmatrix} 1 & 1 \\ \omega & 1+\omega \end{pmatrix}$.

§ 3.2. Characters of Groups

As rich as matrices are, they're a little too bulky. So instead of considering the matrices themselves we consider their traces.

The **trace** of a matrix is the sum of the diagonal components so it's a very easy quantity to calculate –

much easier than determinants or eigenvalues. But it's closely related to eigenvalues in that the **trace of a matrix is the sum of the eigenvalues**. And **similar matrices have the same trace**.

The **character** over a field F of a representation ρ of a finite group G is the map χ: G → F defined by:

$$\chi(g) = \text{trace } \rho(g).$$

Example 5:
The following is one of the representations of S_3 we considered earlier:

	I	**(123)**	**(132)**	**(12)**	**(13)**	**(23)**
ρ	$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$	$\begin{pmatrix}\omega & 0\\0 & \omega^2\end{pmatrix}$	$\begin{pmatrix}\omega^2 & 0\\0 & \omega\end{pmatrix}$	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & \omega^2\\\omega & 0\end{pmatrix}$	$\begin{pmatrix}0 & \omega\\\omega^2 & 0\end{pmatrix}$

Its character is the function given by the following table:

	I	**(123)**	**(132)**	**(12)**	**(13)**	**(23)**
χ	2	–1	–1	0	0	0

Remember that $1 + \omega + \omega^2 = 0$.

Concepts such as 'degree', 'faithful' and 'trivial' extend to characters. So the regular character of S_3 is the character of the regular representation, ρ_7, in example 2. It is:

	I	**(123)**	**(132)**	**(12)**	**(13)**	**(23)**
χ	6	0	0	0	0	0

Theorem 2:
(1) Equivalent representations have the same character.
(2) Characters are constant on conjugacy classes.
Proof: Both of these follow from the fact that similar matrices have same trace. For example if the representation ρ is equivalent to σ then there exists an invertible matrix *S* such that
$\rho(g) = S^{-1}\sigma(g)S$ and, being similar, these have the same trace. 🖑☺

We can easily read off the degree of a character (meaning the degree of the corresponding representation) by simply looking at its value on 1.

Theorem 3: The degree of a character χ is χ(1).
Proof: If ρ is the representation of degree *n* that corresponds to the character χ then ρ(1) is the $n \times n$ identity matrix whose trace is *n*. 🖑☺

There's a shortcut we can use for permutation representations. We can pass directly from the permutations to the character without having to think about the matrices.

Theorem 4: If χ is a permutation character, $\chi(g)$ is the number of symbols fixed by g.

Proof: If ρ is the permutation representation itself then the i-j entry of $\rho(g)$ is 1 if $g(i) = j$ and it is 0 otherwise so $\chi(g)$ is simply the number of 1's on the diagonal. ✋☺

Example 6: If $G = S_4$ and χ is the permutation character, $\chi((123)) = 1$, since (123) fixes 1 symbol, $\chi((12)) = 2$, $\chi(I) = 4$ and $\chi((1234)) = 0$.

Theorem 5: If χ is a character of G over $\mathbb{C}$ of degree n and $g \in G$ has order m then $\chi(g)$ is a sum of n numbers, each of which is an m^{th} root of 1.

Proof: If $g^m = 1$ and ρ is the corresponding representation then $\rho(g)^m$ is the $n \times n$ identity matrix I. The matrix $\rho(g)$ is thus an $n \times n$ matrix and so has n eigenvalues over **C**. Each of these must satisfy the equation $\lambda^m = 1$ and so be an m^{th} root of 1.

Example 7: If g is an element of order 2 and χ is a character of degree 3, corresponding to the representation ρ, then the eigenvalues of $\rho(g)$ will be ± 1. So $\chi(g) \in \{3, 1, -1, -3\}$. If g has order 3 and χ is a character of degree 2, corresponding to the representation ρ, then the eigenvalues of $\rho(g)$ will be two values chosen from 1, ω, ω^2, with possible repetitions. The possibilities for $\chi(g)$ are thus -1, 2, 2ω, $2\omega^2$, $-\omega$, $-\omega^2$.

Note that $1 + \omega = -\omega^2$, $1 + \omega^2 = -\omega$ and $\omega + \omega^2 = -1$.

Theorem 6: The characters, over ℂ, of an element of finite order and its inverse are complex conjugates.
Proof: The eigenvalues of $\rho(g^{-1})$ are the inverses of those for $\rho(g)$. But these eigenvalues are roots of unity and so lie on the unit circle. Hence their inverses are the same as their conjugates. And the sum of these conjugates is the conjugate of the sum. ✋☺

Example 8: If χ is a character of a group of permutations and $\chi((1234)) = 1 + 3i$ then
$\chi((1432)) = 1 - 3i$.

Theorem 7: If χ is the character of a representation, ρ, of a finite group G over ℂ of degree n then $|\chi(g)| \le n$ for all $g \in G$.
Proof: If the eigenvalues of $\rho(g)$ are $\lambda_1, \lambda_2 \ldots, \lambda_n$ then $|\chi(g)| = |\lambda_1 + \lambda_2 + \ldots + \lambda_n|$

$$\le \sum_{i=1}^{n} |\lambda_i| = n.$$ ✋☺

Theorem 8: $\chi(g) = \deg \chi$ if and only if $g \in \ker(\rho)$.
Proof: Let $n = \deg \chi$. The sum of n roots of 1 is equal to n if and only if they're all 1.
If ρ is a corresponding representation then $\rho(g)$, is diagonalisable with all its eigenvalues equal to 1 and so must be I. ✋☺

Example 9: The characters of the above representations ρ_1 to ρ_7 of S_3 are:

	I	(123)	(132)	(12)	(13)	(23)
χ_1	1	1	1	1	1	1
χ_2	1	1	1	–1	–1	–1
χ_3	2	–1	–1	0	0	0
χ_4	2	2	2	0	0	0
χ_5	2	–1	–1	0	0	0
χ_6	3	0	0	1	1	1
χ_7	6	0	0	0	0	0

Example 10: The characters of S_4 include the following. (Since all permutations with a given cycle structure are conjugate they have the same characters, so we need only list the characters by cycle structure.)

I	(××)	(×××)	(××××)	(××)(××)	
1	1	1	1	1	trivial
1	–1	1	–1	1	odd/even
4	2	1	0	0	permutation
24	0	0	0	0	regular

§ 3.3. Class Functions

A **class function** for G over a field F is a map: $G \to F$ which is constant on conjugacy classes.

Example 11: Some class functions for S_3 are:

I	**(12)**	**(13)**	**(23)**	**(123)**	**(132)**
17	–5	–5	–5	π	π
–42	¾	¾	¾	1+i	1+i
1	1	1	1	1	1

Theorem 9: The set of class functions of a group G over a field F is a vector space **CF(G, F)** over F and its dimension over F is the number of conjugacy classes of G.

Proof: It's easily checked that the class functions form a vector space under the usual operations. A basis is the set of class functions, which take the value 1 on some conjugacy class and 0 on the others. ✋☺

Example 12: A basis for the space of class functions of S_3 over $\mathbb{C}$ is:

	I	**(12)**	**(13)**	**(23)**	**(123)**	**(132)**
$\mathbf{e_1}$	1	0	0	0	0	0
$\mathbf{e_2}$	0	1	1	1	0	0
$\mathbf{e_3}$	0	0	0	0	1	1

The class functions given in example 10 are expressible (uniquely) as linear combinations of e_1, e_2, e_3 as: $17e_1 - 5e_2 + \pi e_3$, $-42e_1 + \frac{3}{4}e_2 + (1 + i)e_3$ and $e_1 + e_2 + e_3$.

A character χ is **reducible** if $\chi = \Psi + \Omega$ for some characters Ψ, Ω. If not, it is **irreducible**. Irreducible characters are the basic building blocks of group characters.

Theorem 10: Linear characters are irreducible.
Proof: If $\chi = \Psi + \Omega$ for characters Ψ and Ω, $\deg \chi = \deg \Psi + \deg \Omega \geq 2$. ✋☺

Theorem 11: Every character is a sum of irreducible characters.
Proof: We prove this by induction on the degree of a character. If χ is reducible, $\chi = \Psi + \Omega$ for characters $\chi = \Psi$, Ω. By induction, each is a sum of irreducible characters and hence so too is χ. ✋☺

Certainly if a character is linear we know that it's irreducible. But there are irreducible characters of larger degree. For example χ_3 in example 8 is irreducible. How can we know this? After all it can be broken up as the sum of the two class functions Ψ and Ω.

	I	(123)	(132)	(12)	(13)	(23)
χ_3	2	–1	–1	0	0	0
Ψ	1	–1 + i	–1 + i	1	1	1
Ω	1	–i	–i	–1	–1	–1

How do we know that Ψ and Ω aren't characters? That's not difficult because if Ψ was a character –1 + i would have to be a cube root of 1.

But how do we know that there isn't some other decomposition in which the pieces are both characters? The answer is to make the space of class functions into an inner product space.

From now on we'll be doing what's called **ordinary representation theory**. This simply means that the field over which we operate is ℂ, the field of complex numbers. One can do representation theory over other fields but sometimes things don't go as nicely as they do over ℂ. There are three reasons. Finite fields involve primes that can give problems if they divide the group order. In ℂ no element (except the identity) has finite additive order, or to use technical terminology, ℂ has 'characteristic zero'. But ℝ and ℚ also have characteristic zero. What's wrong with them? Their trouble is that they're not algebraically closed. We may get matrices that fail to have eigenvalues in ℝ or in ℚ, which makes life more complicated. The third reason why ℂ works so beautifully is that we can exploit complex conjugates.

We make CF(G, ℂ) into an inner product space by defining the inner product of two class functions by

$$\langle \chi \mid \Psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\Psi(g)} .$$

Example 13: In the above table all three rows represent class functions.

$\langle \chi_3 \mid \Psi \rangle = \frac{1}{6}$ [2 + 2(–1)(–1–i) + 0] = $\frac{1}{6}$ (4 + 2i) and

$\langle \Omega | \Omega \rangle = \frac{1}{6}$ [1 + 2(–i)(i) + 3(–1)(–1)] = 1.

§ 3.4. The Fundamental Theorem of Characters

Theorem 12: (FUNDAMENTAL THEOREM OF CHARACTERS) The irreducible characters of a finite group G over ℂ form an orthonormal basis for CF(G, ℂ).

Proof: This is proved in Chapter 6. ✋

Theorem 13: The number of irreducible characters of a finite group *G*, over ℂ is equal to the number of conjugacy classes of G.

Proof: We saw already that the dimension of CF(G, ℂ) over ℂ is the number of conjugacy classes. ✋☺

Theorem 14: If χ, Ψ are distinct irreducible characters of a finite group G over ℂ:

$$\sum_{g \in G} \chi(g) \overline{\Psi(g)} = 0 .$$

Proof: Distinct irreducible characters are orthogonal class functions. ✋☺

Theorem 15: If χ is an irreducible character of a finite group G over ℂ then:

$$\sum_{g\in G}|\chi(g)|^2 = |G| .$$

Proof: Irreducible characters have unit length. ✋☺

Theorem 16: Suppose *G* is a finite group with irreducible characters $\chi_1, \ldots , \chi_k$ over ℂ.
If χ is any character, expressible as a sum of irreducible characters by $\chi = \Sigma m_i\chi_I$, then:

(1) for each *i*, $m_i = \langle\chi \mid \chi_i\rangle$;

(2) $\langle\chi \mid \chi\rangle = \Sigma m_i^2$.

Proof: (1) $\langle\chi \mid \chi_i\rangle = \Sigma m_j\langle\chi_j \mid \chi_i\rangle = m_i\langle\chi_I \mid \chi_i\rangle = m_i$.
(2) $\langle\chi|\chi\rangle =$

$$\sum_{i,j=1}^{n}\left\langle\chi_i \mid \chi_j\right\rangle = \sum_{i,j=1}^{k} m_i m_j\left\langle\chi_i \mid \chi_j\right\rangle = \sum_{i=1}^{k} m_i^{\ 2}\left\langle\chi_i \mid \chi_i\right\rangle = \sum_{i=1}^{k} m_i^{\ 2} .$$ ✋☺

Corollary: A character χ is irreducible if and only if

$$\langle\chi \mid \chi\rangle = 1.$$

Theorem 17: If Φ is the regular character and $\chi_1, \ldots , \chi_k$ are the irreducible characters with degrees $n_1, \ldots ,n_k$ then

$$\Phi = \sum_{i=1}^{k} n_i\chi_i .$$

Proof: $\Phi(g) = |G|$ if $g = 1$ and 0 otherwise. So $\langle\Phi|\chi_i\rangle = n_i$.
✋☺

Theorem 18: If $\chi_1, \ldots, \chi_k$ are the irreducible characters with degrees $n_1, \ldots, n_k$ then

$$\sum_{i=1}^{k} n_i^2 = |G|.$$

Proof: If Φ is the regular character,

$$\langle \Phi | \Phi \rangle = |G| = \sum_{i=1}^{k} n_i^2 .$$ ✋☺

§ 3.5. Character Tables

The **character table** for a finite group G, over $\mathbb{C}$, gives the value of each irreducible character on each conjugacy class. We'll use the following notation for character tables:

k is the number of conjugacy classes (also the number of irreducible characters);
$\Gamma_1, \Gamma_2, \ldots, \Gamma_k$ are the conjugacy classes;
Γ^{-1} is the conjugacy class containing the inverses of the elements of Γ;
$\chi_1, \chi_2, \ldots, \chi_k$ are the irreducible characters;

$\overline{\theta}$ is the conjugate of the character θ ;
$n_1, n_2, \ldots, n_k$ are the degrees of the irreducible representations;
$h_1, h_2, \ldots, h_k$ are the sizes of the conjugacy classes;
$m_1, m_2, \ldots, m_k$ are the orders of the elements of each conjugacy class.

The main part of the character table is the $k \times k$ matrix (χ_{ij}) where $\chi_{ij} = \chi_i(\Gamma_j)$.
It is supplemented by explanatory rows listing:

- the conjugacy class names;
- the elements of each class (or a representative);
- the orders of the elements.

The order of the rows and columns in a character table are theoretically arbitrary, but the following conventions are usually used:

(1) The identity conjugacy class {1} is placed first.

(2) Conjugacy classes that are inverses to one another are placed together.

(3) The trivial character is placed first. Then come all the linear characters. Often characters are sorted in ascending order of their degrees, though frequently they're placed in the order in which they're found.

(4) Irreducible characters that are conjugate to each other should be placed adjacent to each other. So often in a character table there will be blocks of the form:

z	$\bar{z}$
$\bar{z}$	z

So a typical character table will look like this.

class	Γ_1	Γ_2	...	Γ_k
elements	**1**	…	…	…
size	**1**	h_2	...	h_k
χ_1	1	1	...	1
χ_2	n_2	χ_{22}	...	χ_{2k}
...	...	...	...	...
χ_k	n_k	χ_{k2}	...	χ_{kk}
order	**1**	m_2	...	m_k

Theorem 19: The character table of a finite group G, over $\mathbb{C}$, has the following properties:

(1) $\sum_{i=1}^{k} h_i = |G|$;

(2) $\sum_{i=1}^{k} n_i^2 = |G|$;

(3) $\sum_{t=1}^{k} h_t \chi_{it} \overline{\chi_{jt}} = \begin{cases} 0 \ if \ i \neq j \\ |G| \ if \ i = j \end{cases}$ (orthogonality of the rows);

(4) $\sum_{t=1}^{k} \chi_{ti} \overline{\chi_{tj}} = \begin{cases} 0 \ if \ i \neq j \\ \dfrac{|G|}{h_i} \ if \ i = j \end{cases}$ (orthogonality of the columns).

(Here χ_{ij} is the value of the irreducible character χ_i on the elements of the conjugacy class Γ_j,

n_i is the degree of χ_i and

h_j is the size of Γ_j.)

Proof:
(1) is just the class equation;
(2) is theorem 17;
(3) is the fundamental theorem of characters
(Theorem 11).

If A is the matrix (a_{ij}) where $a_{ij} = \sqrt{h_j/|G|}\,\chi_{ij}$ then (3) implies that A is a unitary matrix, that is AA* = I where A* is the conjugate transpose of A.
It follows that A*A = I which gives (4). ✋☺

Example 14: The following is the character table for a certain group G.

class	Γ_1	Γ_2	Γ_3	Γ_4
size	**1**	**3**	**4**	**4**
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	–1	0	0
order	**1**	**2**	**3**	**3**

Here ω and ω^2 are the two non-real cube roots of unity. Remember that they're conjugates of one another. And never forget that $1 + \omega + \omega^2 = 0$. That often comes in handy!

We can see that |G| = 1 + 3 + 4 + 4 = 12.
Note too that $12 = 1^2 + 1^2 + 1^2 + 3^2$.

G has 3 elements of order 2 (in Γ_2) and 8 of order 3 (in Γ_3 and Γ_4).

Note the orthogonality of the rows. For example with χ_2 and χ_3 we get:

$$1.1 + 3.1.1 + 4.\omega.\omega + 4.\omega^2.\omega^2 = 4 + 4\omega^2 + 4\omega = 0.$$

And the sums of squares of the moduli of the entries along each row (suitably weighted by the class sizes) are all 12. Along the second row it is $1^2 + 3(1^2) + 4|\omega|^2 + 4|\omega^2| = 12$ and

along the third row it is $3^2 + 3.1^2 + 4.0 + 4.0 = 12$.

Note the orthogonality of the columns. For example taking the 3rd and 4th columns we get $1.1 + \omega.\omega + .\omega^2.\omega^2 = 1 + \omega^2 + \omega = 0$. Taking the sum of squares of the moduli down each column you get the order of the group, 12, divided by the class size.

For example, down column 2 we get $1 + 1 + 1 + 1 = 4 = 12/3$ and down column 4 we get

$1 + 1 + 1 + 0 = 3 = 12/4$.

he '-1' entry is the trace of a 3×3 matrix. This is the sum of the three eigenvalues. Now each of these eigenvalues must be ± 1 since the elements of Γ_2 have order 2. So we can infer that the eigenvalues are $1, -1, -1$. The zero entries in the last row are each the sum of 3 cube roots of unity. The only way to get a zero sum from 3 cube roots of unity is to take exactly one of each. So we can infer that the 3×3 matrices that arise here must have distinct eigenvalues 1, ω and ω^2.

χ_1 is clearly the trivial character. The regular character: [12, 0, 0, 0] is expressible as a sum of irreducible characters as $\chi_1 + \chi_2 + \chi_3 + 3\chi_4$.

As can be seen a considerable amount of information about the group (and the representations themselves) can be recovered from the character table. Of course one has to know something about the group in the first place to be able to construct the character table. But we can learn new things about a group by using characters.

§3.6. The 3N Test for Class Equations

Lemma: If $z = a + b\omega$ where $a, b \in \mathbb{Z}$ then $|z|^2 \in \mathbb{Z}$.
Proof: Multiplying z by its conjugate we get

$$|z|^2 = (a + b\omega)(a + b\omega^2)$$
$$= a^2 + b^2 + ab\omega + ab\omega^2$$
$$= a^2 + b^2 - ab \in \mathbb{Z}.$$

Theorem 6 (3N Test): Suppose $|G| = 3N$ and G has precisely 2 classes of size N.
Then $|G'| = N$ and the class equation for G is

$$3N = 1 + 3t_1 + 3t_2 + \ldots + 3t_k + N*2 \text{ where}$$

$$N = 1 + t_1*3 + t_2*3 + \ldots + t_k*3$$

is the class equation for G′.

Proof: The elements of Γ have order 3 and commute only with their powers.
Suppose Γ is a class of size N.
If $\Gamma = \Gamma^{-1}$ then there exists $g \in \Gamma$ and $x \in G$ such that

$$x^{-1}gx = g^{-1}.$$

Then $x^2 \in C_G(g)$.
But $|C_G(g)| = 3$ so $x \in C_G(g)$.
Hence $g^2 = 1$, a contradiction.
Hence the two conjugacy classes of size N are Γ and Γ^{-1}.
By column orthogonality there must be a non-real entry in the Γ column of the character table for G and, since the eigenvalues of the corresponding matrix must be cube roots of 1, this entry must have the form $a + b\omega$ where a, b are integers and $b \neq 0$.

The character table for G contains the sub-table:

class	**1**	**Γ**	$\mathbf{\Gamma^{-1}}$
size	**1**	**N**	**N**
χ_1	1	1	1
χ_2	n	$a + b\omega$	$a + b\omega^2$
χ_3	n	$a + b\omega^2$	$a + b\omega$
order	**1**	**3**	**3**

where $a, b \in \mathbb{Z}$ with $b \neq 0$.

By the lemma, $|a + b\omega|^2 = |a + b\omega^2|^2$ are positive integers and since the sum of squares of the entries in each of the last two columns is 3N/N = 3, we must have $|a + b\omega|^2 =$

$|a + b\omega^2|^2 = 1$ and all other entries in these columns must be zero.

So, by orthogonality with the first column,

$$\begin{aligned} 0 &= 1 + n(a + b\omega) + n(a + b\omega^2) \\ &= 1 + n(2a - b). \end{aligned}$$

Hence $n = 1$.
Thus G has at least 3 linear characters.

But if χ is any linear character then $\chi(\Gamma) \neq 0$ and so G has exactly 3 linear characters and so $|G'| = N$.
Clearly $G' = G - \Gamma - \Gamma^{-1}$.

The elements of $\Gamma + \Gamma^{-1}$ have centralisers of order 3 so can't commute with any non-trivial element of G′.

Hence if $1 \neq g \in G'$, $C_{G'}(g) = C_G(g)$.
Thus if $g \in G'$ has $t > 1$ conjugates in G′ then it has $3t$ conjugates in G.

Example 8: If G has class equation 48 = 1 + 3 + 12 + 16 + 16 then G′ has class equation
16 = 1 + 1 + 1 + 4 + 4 + 4, which fails the Z Test.

EXERCISES FOR CHAPTER 3

EXERCISE 1:
Examine the following character table for a finite group G and answer the following questions. Give adequate reasons for your answers.

	Γ_1	Γ_2	Γ_3	Γ_4
χ_1	1	1	1	1
χ_2	3	–1	0	0
χ_3	1	1	ω	ω^2
χ_4	1	1	ω^2	ω

(a) What is |G|?
(b) Find the sizes of the conjugacy classes.
(c) Find the orders of the kernels of each of the corresponding irreducible representations.
(d) Which of the irreducible characters are faithful?
(e) Find the order of the elements in each conjugacy class.

EXERCISE 2: Complete the following character table, giving brief explanations as to how each entry is obtained.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
χ_1					
χ_2	1	i	1		–1
χ_3		0	–1	0	0
χ_4	1		1	–1	1
χ_5	1	–i	1		–1

EXERCISE 3: For the character table obtained in exercise 2, compute the size and the order of the elements of each of the conjugacy classes.

EXERCISE 4: Examine the following character table for a finite group G and answer the following questions. Give adequate reasons for your answers.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7
χ_1	1	1	1	1	1	1	1
χ_2	1	1	1	1	-1	-1	-1
χ_3	1	1	ω	ω^2	1	ω	ω^2
χ_4	1	1	ω^2	ω	1	ω^2	ω
χ_5	1	1	ω	ω^2	-1	$-\omega$	$-\omega^2$
χ_6	1	1	ω^2	ω	-1	$-\omega^2$	$-\omega$
χ_7	6	-1	0	0	0	0	0

(a) What is $|G|$?

(b) Find the sizes of the conjugacy classes.

(c) Find the orders of the kernels of each of the irreducible representations.

(d) Which of the irreducible representations are faithful?

(e) Draw the lattice diagram for all the normal subgroups of G.

(f) Find Z(G) and G'. For each of them identify which conjugacy classes they are built up from and give a well-known group that it is are isomorphic to.

(g) Find the order of the elements in each conjugacy class.

(h) Express the following character as a sum of irreducible characters:

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7
χ	14	7	2	2	–6	0	0

EXERCISE 5: Examine the following character table for a finite group G and answer the following questions. Give adequate reasons for your answers.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8	Γ_9
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	–1	–1	1	1
χ_3	–1	2	2	–1	–1	0	0	–1	2
χ_4	1	1	1	–1	1	–1	1	–1	–1
χ_5	1	1	1	–1	1	1	–1	–1	–1
χ_6	–1	2	2	1	–1	0	0	1	–2
χ_7	–2	–2	2	0	2	0	0	0	0
χ_8	1	–2	2	$\sqrt{3}i$	–1	0	0	$-\sqrt{3}i$	0
χ_9	1	–2	2	$-\sqrt{3}i$	–1	0	0	$\sqrt{3}i$	0

(a) How many conjugacy classes does G have?

(b) Which conjugacy class is {1}?

(c) What is |G|?

(d) Find the sizes of the conjugacy classes.

(e) Find the orders of the kernels of each of the irreducible representations.

(f) Which of the irreducible representations are faithful?

(g) Draw the lattice diagram for all the normal subgroups of G.

(h) Find Z(G) and G′. Identify which conjugacy classes they are built up from and describe a well-known group that they are isomorphic to.

(i) How many of the elements of G have order 3?

EXERCISE 6: Complete the following character table, giving brief explanations as to how each entry is obtained.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
size		**3**	**3**	**16**	**3**	**3**		**3**
χ_1			-1			-1		-1
χ_2					1			
χ_3	1		1	ω	1	1		1
χ_4	1		1	ω^2	1	1		1
χ_5	3		$-1+2i$		-1	1		$-1-2i$
χ_6								
χ_7	3		1		-1	$-1-2i$		1
χ_8					-1	$-1+2i$		
order	**1**		**4**	**3**	**2**	**4**		**4**

EXERCISE 7: Complete the following character table, giving brief explanations as to how each entry is obtained.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7
size	**1**	**1**	**6**	**4**	**4**	**4**	**4**
χ_1							
χ_2	1	1	1	ω		ω^2	
χ_3	1	1	1	ω^2			
χ_4	2			$-\omega$			
χ_5	2			$-\omega^2$			
χ_6						1	
χ_7							
order	**1**		**4**	**3**			

SOLUTIONS FOR CHAPTER 3

EXERCISE 1: (a) 12; (b) 1, 3, 4, 4; (c) $|\ker \rho_1| = 12$, $|\ker \rho_2| = 1$, $|\ker \rho_3| = |\ker \rho_4| = 4$; (d) χ_2; (e) 1, 2, 3, 3.

EXERCISE 2:
None of χ_2 to χ_5 are the trivial character so χ_1 must be trivial. Since *i* is not real its conjugate $-i$ must appear in that row, so $\Gamma_4 = \Gamma_2^{-1}$. We can therefore complete columns 2 and 4. By orthogonality of columns 1 and 3 we deduce that deg $\chi_3 = 4$. The character table is thus:

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5
χ_1	1	1	1	1	1
χ_2	1	i	1	–i	–1
χ_3	4	0	–1	0	0
χ_4	1	–1	1	–1	1
χ_5	1	–i	1	i	–1

EXERCISE 3: The group has order 20. We can now compute the sizes of the conjugacy classes: 1, 5, 4, 5, 5. Since the order is even the group must contain elements of order 2. Their characters must be real so the elements of order 2 must lie in Γ_3 or Γ_5 or both. But the centraliser of an element in Γ_3 has order 5, so the elements of order 2 must lie in Γ_5. Also since the group order, 20, is divisible by 5 there must be elements of order 5. Clearly these can't lie in Γ_2 or Γ_4 since i has order 4. So they must lie in Γ_3.

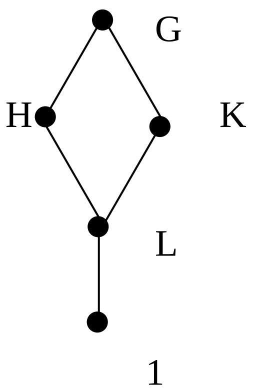
Of course the only element of Γ_1 has order 1, so that just leaves Γ_2 and Γ_4. Since $\Gamma_4 = \Gamma_2^{-1}$ they must all have the same order. This must divide 20 and, since i has order 4, their order must be divisible by 4. Thus they have order exactly 4. The orders of the elements of the conjugacy classes are thus 1, 4, 5, 4, 2 respectively.

EXERCISE 4:

(a) 42; (b)1, 6, 7, 7, 7, 7, 7

(c) 42, 21, 14, 7, 7, 1; (d) χ_7

(e)



$H = \Gamma_1 + \Gamma_2 + \Gamma_5$, $K = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$, $L = \Gamma_1 + \Gamma_2$

(f) $Z(G) = 1$, $G' = L \cong C_7$

(g) Since $L \cong C_7$ the elements of Γ_2 have order 7.
Since $|G| = 42$ there must be elements of orders 2, 3.
The only class that could contain elements of order 2 is Γ_5. By considering the linear characters we see that the order of the elements of Γ_6, Γ_7 is a multiple of 6.

The only multiples of 6 dividing 42 are 6 and 42 and G is clearly not cyclic. So the elements of Γ_6, Γ_7 must have order 6 leaving the elements of Γ_3, Γ_4 being the ones of order 3.
(Note that since $\Gamma_4 = \Gamma_3^{-1}$ and $\Gamma_7^{-1} = \Gamma_6$ the elements in each of these pairs of conjugacy class have the same order.)
The elements of the Γ_i have orders

1, 7, 3, 3, 2, 6, 6, respectively.

(h) $\chi = m_1\chi_1 + ... + m_7\chi_7$ where $m_i = \langle\chi|\chi_i\rangle$ so
$\chi = \chi_1 + 3\chi_2 + 2\chi_5 + 2\chi_6 + \chi_7$.

EXERCISE 5:
(a) 9 conjugacy classes;

(b) Γ_3 (largest modulus);

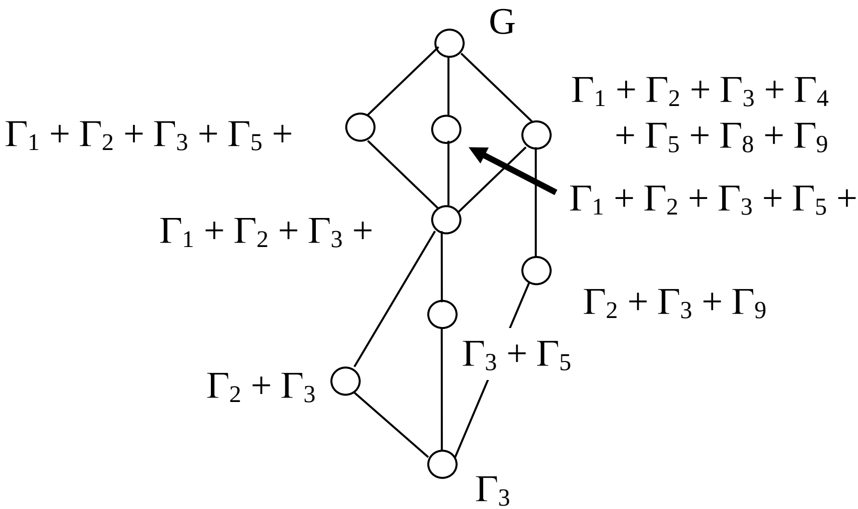
(c) $|G| = \Sigma n_i^2 = 24$.

(d) $|\Gamma_1| = |\Gamma_4| = |\Gamma_5| = |\Gamma_8| = |\Gamma_9| = 24/12 = 2$;
$|\Gamma_2| = |\Gamma_3| = 24/24 = 1$; $|\Gamma_6| = |\Gamma_7| = 24/4 = 6$.
(Check: the sizes total 24.)

(e) $|\ker \rho_1| = 24$;
$|\ker(\rho_2)| = |\Gamma_1| + |\Gamma_2| + |\Gamma_3| + |\Gamma_4| + |\Gamma_5| + |\Gamma_8| + |\Gamma_9| = 12$;
$|\ker(\rho_3)| = 4$; $|\ker(\rho_4)| = |\ker(\rho_5)| = 12$; $|\ker(\rho_6)| = 2$; $|\ker(\rho_7)| = 3$; $|\ker(\rho_8)| = |\ker(\rho_9)| = 1$.

(f) Only ρ_8 and ρ_9 are faithful.

(g)



(h) Z(G) is the union of all the classes of size 1 and consists of classes 2, 3. Z(G) $\cong C_2$.
G' is the intersection of the kernels of the linear representations and consists of classes 1, 2, 3 and 5. It has order 6 but is clearly not isomorphic to S_3 (it has a normal subgroup of order 2) so it is isomorphic to C_6.

(i) For an element *g* of order 3 the only possible eigenvalues for ($\rho(g)$) are 1, ω and ω^2. Hence these are the only possible values for linear characters χ. By inspecting the table we see that the only possibilities for elements of order 3 are Γ_1, Γ_2, Γ_3 and Γ_5. But $\Gamma_3 = \{I\}$ and the elements of Γ_2 are in a subgroup of order 2. Now Γ_5, being in a subgroup of order 3 must consist of 2 elements of order 3. These account for all the elements of order 3 in $G' \cong C_3$ and so Γ_1 must consist of the 2 elements of order 6.

EXERCISE 6: The conjugates of χ_5 must be χ_6 and the conj of χ_7 must be χ_8. The character χ_2 must be the trivial character. $\Gamma_4^{-1} = \Gamma_7$; $|\Gamma_1| = 1$, $|G| = 48$, deg $\chi_1 = 3$, so the remaining entries in Γ_4, Γ_7 are 0, $\Gamma_6^{-1} = \Gamma_2$, $\chi_1(\Gamma_5) = 3$ by orthogonality with Γ_1.

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7	Γ_8
size	**1**	**3**	**3**	**16**	**3**	**3**	**16**	**3**
χ_1	3	–1	–1	0	3	–1	0	–1
χ_2	1	1	1	1	1	1	1	1
χ_3	1	1	1	ω	1	1	ω^2	1
χ_4	1	1	1	ω^2	1	1	ω	1
χ_5	3	1	–1+2i	0	–1	1	0	–1–2i
χ_6	3	1	–1–2i	0	–1	1	0	–1+2i
χ_7	3	–1+2i	1	0	–1	–1–2i	0	1
χ_8	3	–1–2i	1	0	–1	–1+2i	0	1
order	**1**	**4**	**4**	**3**	**2**	**4**	**3**	**4**